Advanced Studies in Contemporary Mathematics
32 (2022), No. 1, pp. 103 - 111
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2022.32.1.103

CENTRAL LAH NUMBERS AND CENTRAL LAH-BELL NUMBERS

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ABSTRACT. The central factorial numbers of the second kind appear in the expansion of powers of x in terms of the central factorials. Central factorial numbers are equally important as Stirling numbers and often arise in properties and applications for difference calculus, spline theory, and approximation theory, etc. This paper begins with the question of the form central Lah numbers and central Lah-Bell numbers comprise of, based upon our observations of the relation between central factorial numbers and the central Bell polynomials. With in this mind, the central Lah numbers and the central Lah-Bell numbers are introduced parallel to the central factorial numbers of the second kind and central Bell polynomials. We investigate several combinatorial identities of these numbers including the generating functions, explicit formulas, binomial convolutions, and represented by the Riemann integral, respectively.

1. INTRODUCTION

The central factorial numbers of the first and second kinds were introduced by Riordan [21]. These factorial numbers consist of the same kind of reciprocity as the identities that hold for the Stirling numbers of the first kind and second kinds. Moreover, it is important to understand that central factorial numbers of the second kind arise in the expansion of powers of x in relation to the central factorials (4); and that they are of equivalent importance to Stirling numbers, frequently occurring in their properties and applications to difference calculus, spline theory, and to approximation theory, etc [1-4, 6, 9-11, 13, 19-21, 23]. The unsigned Lah-number (sometimes called Stirling numbers of the third kind) counts the number of partitions of a set with $\{1,2,\cdots,n\}$ elements into k ordered blocks with no box left empty. It is also important to note that the lah numbers arise within several branches of mathematics, such as non-crossing partitions, Dyck paths, q-analogues as well as falling and rising factorials [3, 5, 7, 8, 14, 15, 17, 18, 22].

This paper begins with the question of what form central Lah numbers and central Lah-Bell numbers comprise of, based upon our observations of the relation between central factorial numbers and the central Bell polynomials. This paper both introduces the central Lah numbers and the central Lah-Bell numbers respectively as 'central' analogues for central factorial numbers of the second kind and central Bell polynomials. Some identities related to these numbers including the generating functions, explicit formulas, binomial convolutions are derived. We also show that each of these two numbers are expressed as Riemann integral, respectively.

First, definitions and preliminary properties required in this paper are introduced.

For $n \geq k \geq 0$, the Stirling numbers of the second kind $S_2(n,k)$ are the numbers of ways to partition a set with n elements into k non-empty subsets, and

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \ (n \geq 0), \quad (\text{see } [3, 16]),$$

where $(x)_0 = 1$ and $(x)_n = x(x-1)\cdots(x-n+1)$.

2010 *Mathematics Subject Classification.* 11F20; 11B68; 11B83 .

Key words and phrases. Lah numbers; Lah-Bell numbers; Lah-Bell polynomials; Central factorial numbers of the second kind; Central Bell polynomials.

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The n-th Bell number B_n $(n \geq 0)$ is the number of ways to partition a set with n elements into non-empty subsets. Bell numbers are a crucial role in enumerative combinatorics. The n-th Bell number and the n-th Bell polynomial are given by generating functions, respectively

$$e^{e^t-1} = \sum_{n=k}^{\infty} B_n \frac{t^n}{n!}, \quad \text{and} \quad e^{x(e^t-1)} = \sum_{n=k}^{\infty} B_n(x) \frac{t^n}{n!}, \ (n \geq 0), \quad (\text{see [3, 16]}), \tag{1}$$

where $B_n = \sum_{k=0}^{n} S_2(n,k)$, and $B_n(x) = \sum_{k=0}^{n} S_2(n,k) x^k \frac{t^n}{n!}$.

It is well-known that

$$(1-t)^{-m} = \sum_{l=0}^{\infty} \binom{-m}{l} (-1)^l t^l = \sum_{l=0}^{\infty} < m >_l \frac{t^l}{l!}, \quad (\text{see [3]}), \tag{2}$$

where $< x >_0 = 1$ and $< x >_n = x(x+1)(x+2)\cdots(x+n-1)$, $(n \geq 1)$.

The central factorials $x^{[n]}$ are given by

(3)

$$x^{[n]} = x(x+\frac{n}{2}-1)(x+\frac{n}{2}-2))\cdots(x-\frac{n}{2}+1) \quad \text{and} \quad x^{[0]} = 1, (n \geq 1), \quad (\text{see } [1-3, 9-12, 21, 23]).$$

Riordan [21] showed that the central factorial numbers of the second kind $T(n,k)$ are given by the coefficients in the expansion of x^n as follows:

$$x^n = \sum_{k=0}^{n} T(n,k) x^{[k]}, \quad (\text{see } [1-3, 9-11, 21]). \tag{4}$$

From (4), it is easy to see that the generating function of $T(n,k)$ is

$$\frac{1}{k!}(e^{\frac{t}{2}} - e^{-\frac{t}{2}})^k = \sum_{n=k}^{\infty} T(n,k) \frac{t^n}{n!}, \quad (n \geq k \geq 0) \quad (\text{see } [1-3, 6-9, 17]). \tag{5}$$

In [10], Kim-Kim introduced the central Bell polynomials $B_n^{(c)}(x)$ are the generating function

$$e^{x(e^{\frac{t}{2}} - e^{-\frac{t}{2}})} = \sum_{n=0}^{\infty} B_n^{(c)}(x) \frac{t^n}{n!}, \quad (\text{see [10]}), \tag{6}$$

where $B_n^{(c)}(x) = \sum_{k=0}^{n} T(n,k) x^k$ and $B_n^{(c)} = B_n^{(c)}(1)$ are the central Bell numbers.

Moreover, they showed that the central factorial Bell numbers represented by Riemann integral.

The unsigned Lah-number $L(n,k)$ counts the number of partitions of a set with n elements into k ordered blocks with no box left empty. The Lah-numbers are rarely called Stirling numbers of the third kind. An explicit formula and the generating function of the Lah numbers $L(n,k)$ are given by respectively

$$L(n,k) = \frac{n!}{k!} \binom{n-1}{k-1}, \quad (\text{see } [3, 5, 7, 8, 12]), \tag{7}$$

and

$$\frac{1}{k!} \left(\frac{t}{1-t} \right)^k = \sum_{n=k}^{\infty} L(n,k) \frac{t^n}{n!}, \quad (\text{see } [3, 5, 7, 8, 12]). \tag{8}$$

From (8), the Lah-Bell numbers LB_n and the Lah-Bell polynomials $LB_n(x)$are given by

$$LB_n = \sum_{k=0}^{n} L(n,k), \ \text{and} \quad LB_n(x) = \sum_{k=0}^{n} L(n,k) x^k \quad (n \geq 0), \quad (\text{see [5, 7, 8]}), \tag{9}$$

respectively. In addition, the generation function of Lah-Bell polynomials $LB_n(x)$ is given by

$$e^{x(\frac{t}{1-t})} = \sum_{n=0}^{\infty} LB_n(x)\frac{t^n}{n!}, \quad (\text{see [5, 7, 8]}). \tag{10}$$

When $x = 1$, $LB_n = LB_n(1)$, which are Lah-Bell numbers.

2. Central Lah numbers and Central Lah-Bell numbers

This section introduces and investigates properties of both central Lah numbers and the central Lah-Bell numbers. We will demonstrate how both of these numbers are ”central” analogues for central factorial numbers of the second kind and central Bell polynomials.

In view of (5), we introduce the generating function of central Lah numbers $L^{(C)}(n,k)$ given by

$$\frac{1}{k!}\left(2\left(\frac{1}{2-t} - \frac{1}{2+t}\right)\right)^k = \sum_{n=k}^{\infty} L^{(C)}(n,k)\frac{t^n}{n!}. \tag{11}$$

Theorem 1. *For $n \geq k \geq 0$, we have*

$$L^{(C)}(n,k) = \sum_{i=0}^{k}\sum_{l=k-i}^{n} (-1)^{k-i+l} 2^{-n} L(n-l,i)L(l,k-i),$$

where $L(0,0) = 1,\ L(n,0) = 0$, and $L(n,k) = 0$ for all $k > n$.

Proof. From (2) and (8), we observe that

$$\begin{aligned}
\sum_{n=k}^{\infty} L^{(C)}(n,k)\frac{t^n}{n!} &= \frac{1}{k!}\left(2\left(\frac{1}{2-t} - \frac{1}{2+t}\right)\right)^k = \frac{1}{k!}\left(\frac{\frac{1}{2}t}{1-\frac{1}{2}t} - \frac{-\frac{1}{2}t}{1-(-\frac{1}{2}t)}\right)^k \\
&= \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}\left(\frac{\frac{1}{2}t}{1-\frac{1}{2}t}\right)^i (-1)^{k-i}\left(\frac{-\frac{1}{2}t}{1-(-\frac{1}{2}t)}\right)^{k-i} \\
&= \sum_{i=0}^{k}(-1)^{k-i}\frac{1}{i!}\left(\frac{\frac{1}{2}t}{1-\frac{1}{2}t}\right)^i \frac{1}{(k-i)!}\left(\frac{-\frac{1}{2}t}{1-(-\frac{1}{2}t)}\right)^{k-i} \\
&= \sum_{i=0}^{k}(-1)^{k-i}\sum_{m=i}^{\infty} L(m,i)\frac{(\frac{1}{2}t)^m}{m!}\sum_{l=k-i}^{\infty} L(l,k-i)\frac{(-\frac{1}{2}t)^l}{l!} \\
&= \sum_{n=k}^{\infty}\left(\sum_{i=0}^{k}\sum_{l=k-i}^{n}(-1)^{k-i+l}\left(\frac{1}{2}\right)^n L(n-l,i)L(l,k-i)\right)\frac{t^n}{n!}.
\end{aligned} \tag{12}$$

By comparing with coefficients of both sides of (12), we attain the desired identity. □

The first few central Lah numbers are

$$L^{(C)}(1,1) = 1, \qquad L^{(C)}(2,1) = \left(\frac{1}{2}\right)^2, \qquad L^{(C)}(2,2) = 3\left(\frac{1}{2}\right)^2,$$

$$L^{(C)}(3,1) = 10\left(\frac{1}{2}\right)^3, \quad L^{(C)}(3,2) = 0, \quad L^{(C)}(3,3) = 4\left(\frac{1}{2}\right)^3, \quad \textit{etc.}$$

Theorem 2. *For $n \geq 1$, an explicit formula of central Lah numbers $L^{(C)}(n,k)$ is*

$$L^{(C)}(n,k) = \frac{1}{k!}\sum_{j=0}^{k}\sum_{l=0}^{n}\binom{k}{j}\binom{n}{l}(-1)^{j+n-l}2^{-n} < j >_{n-l} < k-j >_{l} .$$

Proof. From (2) and (11), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} L^{(C)}(n,k)\frac{t^n}{n!} &= \frac{1}{k!}\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)^{k}\\ &= \frac{1}{k!}2^k\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left(\frac{1}{2+t}\right)^{j}\left(\frac{1}{2-t}\right)^{k-j}\\ &= \frac{1}{k!}2^k\sum_{j=0}^{k}\binom{k}{j}(-1)^j\sum_{d=0}^{\infty}\left(\frac{1}{2}\right)^{k} < j >_d \frac{(-\frac{t}{2})^d}{d!}\sum_{l=0}^{\infty} < k-j >_l \frac{(\frac{t}{2})^l}{l!}\\ &= \sum_{n=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\sum_{l=0}^{n}\binom{k}{j}\binom{n}{l}(-1)^{j+n-l}2^{-n} < j >_{n-l} < k-j >_l \frac{t^n}{n!}.\end{aligned} \tag{13}$$

By comparing with coefficients of both sides of (13), we have desired result. □

Naturally, in view of (9), we define the central Lah-Bell polynomials $LB_n^{(C)}(x)$ by

$$LB_n^{(C)}(x) = \sum_{k=0}^{n} L^{(C)}(n,k)x^k, \quad (n \geq 0), \tag{14}$$

when $x = 1$, $LB^{(C)} = LB^{(C)}(1)$, which are called the central Lah-Bell numbers.

Theorem 3. *For $n \geq k \geq 0$, the generating function of the central Lah-Bell polynomials $LB_n^{(C)}(x)$ is*

$$exp\left(2x\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right) = \sum_{n=0}^{\infty} LB_n^{(C)}(x)\frac{t^n}{n!}.$$

Proof. From (11) and (14), we have

$$\begin{aligned}\sum_{n=0}^{\infty} LB_n^{(C)}(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} L^{(C)}(n,k)x^k\right)\frac{t^n}{n!} = \sum_{k=0}^{\infty} x^k\left(\sum_{n=k}^{\infty} L^{(C)}(n,k)\right)\frac{t^n}{n!}\\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)^{k}x^k = exp\left(2x\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right).\end{aligned} \tag{15}$$

By (15), we get the desired result. □

Corollary 4. *For $n \geq k \geq 0$, the generating function of the central Lah-Bell numbers is*

$$exp\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right) = \sum_{n=0}^{\infty} LB_n^{(C)}\frac{t^n}{n!}.$$

Theorem 5. *For $n \geq 1$, we have*

$$LB_n^{(C)}(x) = \sum_{k=0}^{n}\sum_{m=0}^{n}\binom{n}{m}2^{n+k}(-1)^m < k >_m L(n-m,k)x^k, \quad (n \geq 0).$$

Proof. From (2), (10) and Lemma 4, we observe that

$$
\begin{aligned}
\sum_{n=0}^{\infty} LB_n^{(C)}(x)\frac{t^n}{n!} &= \exp\left(2x\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)\\
&= \sum_{k=0}^{\infty} x^k\frac{1}{k!}\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)^k\\
&= \sum_{k=0}^{\infty} x^k 2^k\left(\frac{1}{2+t}\right)^k\frac{1}{k!}\left(\frac{2+t}{2-t}-1\right)^k\\
&= \sum_{k=0}^{\infty} x^k 2^{2k-k}\left(\frac{1}{1+\frac{t}{2}}\right)^k\frac{1}{k!}\left(\frac{\frac{t}{2}}{1-\frac{t}{2}}\right)^k\\
&= \sum_{k=0}^{\infty} x^k 2^k\sum_{m=0}^{\infty} <k>_m\frac{(-\frac{t}{2})^m}{m!}\sum_{l=k}^{\infty} L(l,k)\frac{(\frac{1}{2}t)^l}{l!}\\
&= \sum_{k=0}^{\infty} x^k 2^k\sum_{n=k}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}<k>_m(-1)^m 2^n L(n-m,k)\right)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{n}\binom{n}{m}x^k 2^{n+k}(-1)^m<k>_m L(n-m,k)\right)\frac{t^n}{n!}.
\end{aligned}
\tag{16}
$$

By comparing with coefficients of both sides of (16), we have the desired result. □

Theorem 6. *For $n\geq 1$, we have*

$$L^{(C)}(n,k)=\frac{2(n!)}{k!\pi}Im\int_0^{\pi}2^k\left(\frac{1}{2-e^{i\theta}}-\frac{1}{2+e^{i\theta}}\right)^k\sin n\theta d\theta.$$

Proof. For $j,k\geq 0$, by (2), we observe that

$$
\begin{aligned}
&Im\int_0^{\pi}\left(\frac{1}{2+e^{i\theta}}\right)^j\left(\frac{1}{2-e^{i\theta}}\right)^{k-j}\sin n\theta d\theta\\
&= 2^{-k}Im\int_0^{\pi}\left(\frac{1}{1-(-\frac{e^{i\theta}}{2})}\right)^j\left(\frac{1}{1-\frac{e^{i\theta}}{2}}\right)^{k-j}\sin n\theta d\theta\\
&= 2^{-k}Im\int_0^{\pi}\sum_{d=0}^{\infty}<j>_d\frac{(-\frac{1}{2}e^{i\theta})^d}{d!}\sum_{l=0}^{\infty}<k-j>_l\frac{(\frac{1}{2}e^{i\theta})^l}{l!}\sin n\theta d\theta\\
&= 2^{-k}Im\int_0^{\pi}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\binom{m}{l}(-1)^{m-l}2^{-m}<j>_{m-l}<k-j>_l\frac{e^{mi\theta}}{m!}\sin n\theta d\theta\\
&= 2^{-k}\sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{m}{l}(-1)^{m-l}2^{-m}<j>_{m-l}<k-j>_l\frac{1}{m!}\int_0^{\pi}\sin m\theta\sin n\theta d\theta\\
&= 2^{-k}\sum_{l=0}^{n}\binom{n}{l}(-1)^{n-l}2^{-n}<j>_{n-l}<k-j>_l\frac{1}{n!}\frac{\pi}{2}.
\end{aligned}
\tag{17}
$$

From (17), we get

$$\begin{aligned}
&\frac{1}{k!}Im\int_0^{\pi}2^k\left(\frac{1}{2-e^{i\theta}}-\frac{1}{2+e^{i\theta}}\right)^k\sin n\theta d\theta\\
&\quad=\frac{1}{k!}Im\int_0^{\pi}2^k\sum_{j=0}^{k}\binom{k}{j}(-1)^j\left(\frac{1}{2+e^{i\theta}}\right)^j\left(\frac{1}{2-e^{i\theta}}\right)^{k-j}\sin n\theta d\theta\\
&\quad=\frac{1}{k!}2^k\sum_{j=0}^{k}\binom{k}{j}(-1)^j Im\int_0^{\pi}\left(\frac{1}{2+e^{i\theta}}\right)^j\left(\frac{1}{2-e^{i\theta}}\right)^{k-j}\sin n\theta d\theta\\
&\quad=\frac{1}{k!}\sum_{j=0}^{k}\sum_{l=0}^{n}\binom{k}{j}\binom{n}{l}(-1)^{j+n-l}2^{-n}<j>_{n-l}<k-j>_l\frac{1}{n!}\frac{\pi}{2}.
\end{aligned}\tag{18}$$

On the other hand, from Theorem 2, we have

$$L^{(C)}(n,k)=\frac{1}{k!}\sum_{j=0}^{k}\sum_{l=0}^{n}\binom{k}{j}\binom{n}{l}(-1)^{j+n-l}2^{-n}<j>_{n-l}<k-j>_l\,.\tag{19}$$

From (18) and (19), we have desired result. □

Theorem 7. *For $n\geq 1$, we have*

$$LB_n^{(C)}=\frac{2(n!)}{\pi}Im\int_0^{\pi}\exp\left(2\left(\frac{1}{2-e^{i\theta}}-\frac{1}{2+e^{i\theta}}\right)\right)\sin n\theta d\theta.$$

Proof. By (14) and Theorem 6, we observe that

$$\begin{aligned}
&Im\int_0^{\pi}\exp\left(2\left(\frac{1}{2-e^{i\theta}}-\frac{1}{2+e^{i\theta}}\right)\right)\sin n\theta d\theta\\
&\quad=\sum_{k=0}^{\infty}\frac{1}{k!}Im\int_0^{\pi}2^k\left(\frac{1}{2-e^{i\theta}}-\frac{1}{2+e^{i\theta}}\right)^k\sin n\theta d\theta\\
&\quad=\sum_{k=0}^{\infty}\frac{\pi}{n!2}L^{(C)}(n,k).
\end{aligned}\tag{20}$$

By comparing with coefficients of both sides of (20), we have desired result. □

Theorem 8. *For $n\geq 1$, we have*

$$\begin{aligned}
\sum_{k=0}^{n}L^{(C)}(n,k)x^{[k]}=&\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{n}\binom{2x}{m}\binom{2x-m+2i}{j}\binom{m}{i}\binom{n}{l}\\
&\times(-1)^{j+n-l}2^{2x-m+2i-n}<2x-m+2i-j>_l<j>_{n-l}\,.
\end{aligned}$$

Proof. It is known that the generating function of central factorials is given by

$$\left(\frac{t}{2}+\sqrt{\frac{1}{4}t^2+1}\right)^{2x}=\sum_{n=0}^{\infty}x^{[n]}\frac{t^n}{n!},\quad \text{(see [13])}.\tag{21}$$

From (21), we observe that

$$
\begin{aligned}
&\left(\frac{1}{2-t}-\frac{1}{2+t}+\sqrt{\left(\frac{1}{2-t}-\frac{1}{2+t}\right)^{2}+1}\ \right)^{2x} \\
&\qquad\qquad = \sum_{k=0}^{\infty} x^{[k]} \frac{\left(2\left(\frac{1}{2-t}-\frac{1}{2+t}\right)\right)^{k}}{k!} \\
&\qquad\qquad = \sum_{k=0}^{\infty} x^{[k]} \sum_{n=k}^{\infty} L^{(C)}(n,k)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} L^{(C)}(n,k) x^{[k]} \frac{t^n}{n!}.
\end{aligned} \tag{22}
$$

On the other hand, by (2), we obtain

$$
\begin{aligned}
&\left(\frac{1}{2-t}-\frac{1}{2+t}+\sqrt{\left(\frac{1}{2-t}-\frac{1}{2+t}\right)^{2}+1}\ \right)^{2x} \\
&= \sum_{m=0}^{\infty}\binom{2x}{m}\left(\frac{1}{2-t}-\frac{1}{2+t}\right)^{2x-m}\left(\left(\frac{1}{2-t}-\frac{1}{2+t}\right)^{2}+1\right)^{\frac{m}{2}} \\
&= \sum_{m=0}^{\infty}\binom{2x}{m}\sum_{i=0}^{\infty}\binom{\frac{m}{2}}{i}\left(\frac{1}{2-t}-\frac{1}{2+t}\right)^{2x-m+2i} \\
&= \sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\binom{2x}{m}\binom{\frac{m}{2}}{i}\sum_{j=0}^{\infty}\binom{2x-m+2i}{j}(-1)^{j}\left(\frac{2}{1-(-\frac{t}{2})}\right)^{j}\left(\frac{2}{1-(\frac{t}{2})}\right)^{2x-m-2i-j} \\
&= \sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{2x}{m}\binom{\frac{m}{2}}{i}\binom{2x-m+2i}{j}(-1)^{j}2^{2x-m-2i} \\
&\qquad\qquad \times \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}(-1)^{n-l}2^{-n} < 2x-m+2i-j >_{l} < j >_{n-l}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{n}\binom{2x}{m}\binom{2x-m+2i}{j}\binom{m}{i}\binom{n}{l}(-1)^{j+n-l}\right. \\
&\qquad\qquad \left. \times 2^{2x-m+2i-n} < 2x-m+2i-j >_{l} < j >_{n-l}\right)\frac{t^n}{n!}.
\end{aligned} \tag{23}
$$

By comparing with coefficients of (22) and (23), we arrive at what we desired. □

3. Conclusion

In conclusion, we successfully introduced the central Lah numbers and the central Lah-Bell numbers as "central" analogues for the central factorial numbers of the second kind and the central Bell polynomials. For each of these numbers, we derived generator functions, explicit formulas, and show that it was expressed as binomial convolutions. Furthermore, by showing that each of these two numbers are expressed as Riemann integral in Theorem 6 and 7, respectively, we can infer approximate values for each of them. As for Theorem 8, we demonstrated the relations between the central Lah-numbers and the central factorials numbers.

As a follow-up study of this study, it was shown in the references [7] that the central Lah numbers and the central Lah-Bell numbers were related to well-known incomplete and complete Bell polynomials. Looking ahead, we now suggest investigating some useful applications for the central

Lah numbers and polynomials, and the central Lah-Bell numbers and polynomials introduced in this paper.

Acknowledgments
The author would like to thank the referees for the detailed and valuable comments that helped improve the original manuscript in its present form.

Availability of data and material
This paper does not use data and material.

Funding
This work was supported by research grants from Daegu Catholic University in 2021.

Ethics approval and consent to participate
The author declare that there is no ethical problem in the production of this paper.

Competing interests
The author declare no conflict of interest.

Consent for publication
The author want to publish this paper in this journal.

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